Lesson 22 1st-order linear D.E.'s - Part II

I. Getting into Standard Form

II. Examples

Friday 10/27 Quiz on 1st-order linear D.E.'s

Reminder

Definitions/Reminders

① A D.E. is a 1st-order linear D.E. if it can be written in the form

y+C=0

line

$$A(x)\frac{dy}{dx} + B(x)y + C(x) = 0$$

Keys: ① y only appears in $\frac{dy}{dx}$ & y

No other ops on y.

The standard form for a 1st-order linear D.E. is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

② Everything else only involves x

I. Getting into Standard Form

- Divide by $A(x)$ so there is a 1 in front of $\frac{dy}{dx}$
- Put $\frac{dy}{dx}$ & y term $(P(x)y)$ on one side

and everything else on other side.

Recall

$$\frac{dy}{dx} + P(x)y = Q(x)$$

① Calculate the integrating factor

$$u(x) = e^{\int P(x)dx}$$

② Solutions look like

$$y\,u(x) = \int u(x)\,Q(x)\,dx$$

③ Solve for y

II. Examples

$\boxed{Ex}$ (Nagle, et. al.)

$$x\left(\frac{1}{x}\frac{dy}{dx}\right) = \left(x\cos(x) + \frac{2y}{x^2}\right)x$$

$$\frac{dy}{dx} = x^2\cos(x) + \frac{2y}{x}$$

$$-\frac{2y}{x} \qquad\qquad -\frac{2y}{x}$$

$$\frac{dy}{dx} - \frac{2y}{x} = x^2\cos(x)$$

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos(x)$$

Std Form: $P(x) = \frac{-2}{x}$ $\qquad Q(x) = x^2\cos(x)$

integrating factor: $u(x) = e^{\int \frac{-2}{x}dx} = e^{-2\ln(x)} = e^{\ln(x^{-2})}$

$$= x^{-2} = \frac{1}{x^2}$$

Sol'n $y\,u(x) = \int u(x)\,Q(x)\,dx$

$$y\cdot\frac{1}{x^2} = \int \frac{1}{x^2}\cdot x^2\cos(x)\,dx$$

$$y\cdot\frac{1}{x^2} = \int \cos(x)\,dx$$

$$y\cdot\frac{1}{x^2} = \sin(x) + C \implies y = x^2\sin(x) + Cx^2$$

$\boxed{Ex}$ (Edwards et. al.)

$$\underline{x\frac{dy}{dx} - 3y} = \frac{x^3}{x} \qquad y(1) = 10$$

x

$$\frac{dy}{dx} - \frac{3}{x}y = x^2$$

$$P(x) = \frac{-3}{x} \qquad Q(x) = x^2$$

<u>integrating factor</u> $u(x) = e^{\int \frac{-3}{x}dx} = e^{-3\ln(x)} = e^{\ln(x^{-3})}$

$$= x^{-3} = \frac{1}{x^3}$$

<u>Sol'n</u>: $u(x)\,y = \int u(x)Q(x)\,dx$

$$\frac{1}{x^3}y = \int \frac{1}{x^3}x^2\,dx$$

$$\frac{1}{x^3}y = \int \frac{1}{x}\,dx$$

$$\frac{1}{x^3}y = \ln(|x|) + C$$

$$y(1) = 10$$

$\underbrace{x=1}_{\text{positive}}$ $\quad y = 10$

so we want the branch of $\ln(|x|)$ where x is positive

$$y = x^3\ln(x) + Cx^3$$

$$10 = 1^3\ln(1) + C(1)^3$$

$$10 = C$$

Since x is positive $\ln(|x|) = \ln(x)$

$$\boxed{y = x^3\ln(x) + 10x^3}$$

$\boxed{Ex}$ (Edwards, et. al.)

$$xy' + 5y = 7x^2 \qquad y(2) = 5$$

$$x\frac{dy}{dx} + 5y = 7x^2$$

$$\frac{dy}{dx} + \frac{5}{x}y = 7x$$

$$u(x) = e^{\int \frac{5}{x}dx} = e^{5\ln(x)} = e^{\ln(x^5)} = x^5$$

$u(x)y = \int u(x)Q(x)dx$: $$x^5 y = \int 7x \cdot x^5 dx$$

$$x^5 y = \int 7x^6 dx$$

$$x^5 y = \frac{7x^7}{7} + C$$

$$x^5 y = x^7 + C$$

$$y = x^2 + \frac{C}{x^5}$$

$$\boxed{y = x^2 + \frac{32}{x^5}}$$

$y(2) = 5$

$$5 = 2^2 + \frac{C}{2^5}$$

$$5 = 4 + \frac{C}{32} \Rightarrow C = 32$$

Ex Find general sol'n. (Edwards et. al.)

$$x^3 \frac{dy}{dx} + 2x^4 y = x^4$$

$$\boxed{y = \frac{1}{2} + \frac{C}{e^{x^2}}}$$

$$\frac{dy}{dx} + 2xy = x$$

$$u(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2} y = \int e^{x^2} x\,dx$$

u-substitution:

$u = x^2$

$$e^{x^2} y = \frac{1}{2} e^{x^2} + c$$

$$y = \frac{1}{2} + \frac{c}{e^{x^2}}$$